Secure and Adaptive State Estimation for PMU-equipped Smart Grid

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***Abstract*—Modern power systems are constantly subjected to various disturbances, device failures, as well as data attacks. To improve the quality of monitoring and control in smart grid, researchers have conducted extensive studies in exploring the advantages of real-time digital meters such as the Phasor Measurement Units, combining with dynamic state estimation methods such as Kalman filters.**

Kalman filters achieve optimal performance only when the system noise characteristics have known statistical properties. However, in practice the noise models are usually difficult to obtain, especially with the existence of malicious data attack. A lightweight and efficient algorithm, namely Adaptive Kalman Filter with Inflatable Noise Variances, is presented in this paper for its ability to identify and reduce the impact of incorrect system models and bad measurement data. Simulations demonstrate its robustness to suboptimal system modeling, sudden changes of system dynamics and bad data injection.

***Index Terms*—Power system state estimation; Phasor Measurement Units (PMU); data security; bad data processing**

I. Introduction

POWER systems are complex and highly dynamic, however, traditional state estimators typically receive measured data from a Supervisory Control And Data Acquisition (SCADA) system in the time interval of several seconds, which is not frequent enough to capture the system dynamics. Nowadays phasor measurement technology has revealed promising capability for tracking power system state in real-time, in conjunction with dynamic state estimation methods such as Kalman filters.

Power system state can be estimated using a hypothetical system model and available measurements. Ideally, the hypothetical model is the "true" model, with accurate noise statistical characteristics. However, in practice, as statistician George Box once said, "All models are wrong, but some are useful" [5]. Besides modeling errors, the measurements can also contain errors, either due to device failure or malicious data attack. Among which, bad data injection is one of the most dangerous attacks in smart grid. The attackers can construct bad data that are evading the bad data detection mechanisms in a power system, leading to energy theft on the end-user side, false dispatch in the distribution process, and device breakdown during power generation [6].

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When the hypothetical model does not match the actual process, and/or the measurements contain bad data, the estimated state can rapidly deviate from the true state. Thus we propose a general approach: an adaptive Kalman filter with inflatable noise variances (AKF with InNoVa), for more robust state estimation. This novel approach features a dual-test procedure: *normalized innovation test* and *normalized residual test*, to help identify modeling errors and measurement errors separately. By adjusting different noise parameters on-the-fly, our algorithm is lightweight, yet remarkably efficient in dealing with inaccurate system models and measurement data errors/attacks.

This paper is organized as follows. Section II introduces the background. Section III briefly describes the problem formulation, presents the principles and implementation details of our adaptive Kalman filter algorithm. Simulation results are given in Section IV, demonstrate the robustness of our algorithm under different abnormal conditions. Section V concludes the paper.

II. Background and Related Work

Traditionally, a weighted least square (WLS) estimator is the classic method for power system state estimation [1]. In [7] the authors studied the use of phasor measurements in WLS state estimation, and how to identify bad data with normalized residual vectors. The WLS estimator is considered "static" because for a certain time instant, the state is only calculated from the measurement set of the same instant of time. On the other hand, if an estimator has the ability of predicting the state ahead of time, it is considered "dynamic". Most dynamic state estimation algorithms are based on the Kalman filter, which is also the focus of our paper here; however a reliable system state evolution model may not be available in practice, and the measurements may contain unexpected errors.

To deal with unpredictable process changes, such as maneuvering target motion, [9] proposed a reverse prediction adaptive Kalman filtering algorithm, which adjusts the process noise parameters to improve filtering precision, assuming the measurement model is correct.

To deal with measurement errors, one approach is to identify and eliminate bad measurements: [8] proposed a robust extended Kalman filter (REKF) which calculates normalized innovation vectors to detect the presence of sudden load change or bad data, and flag the suspected measurements; then it calculates normalized WLS residual vectors to confirm bad measurements. Another approach is to adjust the measurement model: [10] adjusted the measurement noise parameters, so that the robust algorithm can be immunized to the polluted measurements.

Assuming the process and measurement noise statistics are both unknown, the authors of [11] constructed the autocorrelation functions of innovation sequence, and incorporate information about the quality of the autocorrelation parameters through a WLS method. The latter permits to generate a convergent sequence to the steady state filter, which allows to determine process noise parameters and measurement noise parameters after some manipulations.

In this paper, we allow the existence of unexpected process changes as well as data errors. Our proposed method, AKF with InNoVa, is able to adjust noise parameters on-the-fly by analyzing the *innovation* and *residual* vectors. This method is shown to be convenient, effective and robust under various testing scenarios.

III. Adaptive Kalman Filter and its Application

Ideally, the Kalman filter estimates the state by minimizing the estimate error covariance, in a recursive prediction-correction manner. Readers can refer to [12] for a more detailed introduction. The prediction and correction steps are expressed by a set of time update and measurement update equations, respectively:

$$\text{Predict}\begin{cases} \hat{x}_k^- = A\hat{x}_{k-1} + Bu_{k-1} \\ P_k^- = AP_{k-1}A^T + Q \end{cases}$$
$$\text{Correct}\begin{cases} K_k = P_k^- H^T(HP_k^- H^T + R)^{-1} \\ \hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-) \\ P_k = (I - K_k H)P_k^- \end{cases} \tag{1}$$

We define $\hat{x}_k^- \in \mathcal{R}^n$ to be the *a priori* state estimate at time step k given the knowledge of the process prior to k, so $e_k^- \equiv x_k - \hat{x}_k^-$ is called the *a priori* estimate error and $P_k^- \equiv E[e_k^- e_k^{-T}]$ is the *a priori* estimate error covariance. The time update equations are responsible for projecting forward (in time) the previous state x_{k-1} and error covariance estimates P_{k-1} to obtain the *a priori* estimates for the next time step k. Similarly $\hat{x}_k \in \mathcal{R}^n$ is called *a posteriori* state estimate at time step k given measurement z_k, $e_k \equiv x_k - \hat{x}_k$ and $P_k \equiv E[e_k e_k^T]$ are the *a posteriori* estimate error and the *a posteriori* estimate error covariance respectively. The measurement update equations are responsible for the feedback — incorporating a new measurement into the *a priori* estimate to obtain an improved *a posteriori* estimate.

The process noise and measurement noise are assumed to be independent and normally distributed white random variables, with process noise covariance Q and measurement noise covariance R respectively. K is a $n \times m$ matrix called the Kalman gain matrix, reflecting how we trust the actual measurement z_k versus the predicted measurement $H\hat{x}_k^-$. From its expression, one can tell that larger values of R place more weight on the predicted value while smaller values of R place more weight on the measured values.

We define the *innovation vector*, $\mathcal{I}_k^- = (z_k - H\hat{x}_k^-)$, to be the difference between the true measurement vector and that computed from the *a priori* state estimate. The corresponding *innovation covariance* is $S_k = HP_k^- H^T + R$ [14].

Although some authors use the terms "innovation" and "residual" interchangeably, they should not be confused here. Throughout, we refer to $\mathcal{I}_k^- = (z_k - H\hat{x}_k^-)$ as the *innovation*, to distinguish it from the *residual* $\mathcal{I}_k = (z_k - H\hat{x}_k)$, which is the difference between the true measurement vector and that computed from the *a posteriori* state estimate [13].

Lemma 1: Ideally, the *residual vector* $\mathcal{I}_k = (z_k - H\hat{x}_k)$ should be normally distributed, with zero mean and covariance $RS_k^{-1}R$. We define $T_k = RS_k^{-1}R$ as the *residual covariance*.

The proof is proved in Appendix A. Based on the lemma above, we propose our new algorithm: Adaptive Kalman Filter with Inflatable Noise Variances (InNoVa). In this adaptive Kalman filter (AKF) we treat deviations from the true model and measurement errors as un-modeled noise. By definition, these errors are unknown and unpredictable, so they cannot be reflected in the noise covariance matrices R and Q within the original model. For now we do not consider cases with multiple/switching process models.

Traditionally, people examine the *innovation vector* to assess the performance of a filter. It is defined as $\mathcal{I}_k^- = (z_k - H\hat{x}_k^-)$, which should have a normal distribution with zero mean and covariance S_k. When the implemented process model does not match reality, the mean of the innovation can shift, and the magnitude grows, such that eventually the *normalized innovation vector* (normalized by its covariance S_k) exceeds a predetermined threshold. However, it is usually impossible to determine whether the shift/growth is caused by a process model mismatch, a faulty measurement, or both. Note that Q and R are already blended (indistinguishable) in S_k. This is why we want to investigate the *residaul vector* $\mathcal{I}_k = (z_k - H\hat{x}_k)$ as well: its ideal covariance T_k can be used to identify the un-modeled measurement errors.

The basic idea of our approach is as follows. To simplify the notation we omit the time step count k. Within each filtering cycle, after the prediction step, we compute the ideal *innovation covariance* S and the *normalized innovation vector* $\tilde{\mathcal{I}}^-$ where

$$\tilde{\mathcal{I}}_i^- = |\mathcal{I}_i^-|/\sqrt{S_{ii}}, \quad i = 1, 2, ...n. \tag{2}$$

If $\tilde{\mathcal{I}}_i^- > \tau$ for some threshold τ, then $i \in Out$, where Out holds the outlier indices. In our experiments, we used $\tau = 3$ (measurement units).

First, we assume the outliers are caused by unknown process noise, so we want to "inflate" Q by a diagonal matrix ΔQ, such that P^- is also inflated by ΔQ. Thus S is consequently inflated by $\Delta S = H(\Delta Q)H^T$ and

$$\tilde{\mathcal{I}}_i^- = |\mathcal{I}_i^-|/\sqrt{S_{ii} + \Delta S_{ii}} \leq \tau, \quad i = 1, 2, ...n. \tag{3}$$

It is easy to show that

$$\Delta S_{ii} = \sum_{j=1}^{n} H_{ij}^2 \Delta Q_j = (H(i,:) \cdot H(i,:))([\Delta Q_1, ...\Delta Q_n]^T), \tag{4}$$

where "·" denotes dot product. We could simply use the linear programming method to solve an optimization problem:

$$\min \sum_{i=1}^{n} \Delta Q_i \tag{5}$$

$$\begin{aligned} s.t. \quad \Delta S_{ii} &= (H(i,:) \cdot H(i,:))([\Delta Q_1, ...\Delta Q_n]^T) \\ &\geq (|\mathcal{I}_i^-|/\tau)^2 - S_{ii}, \quad \forall i \in Out \\ \Delta Q_1 &\geq 0, \Delta Q_2 \geq 0, ...\Delta Q_n \geq 0. \end{aligned}$$

The inflated Q is then incorporated into the correct step. Similarly, we compute the ideal *residual covariance* T and the *normalized residual vector* $\tilde{\mathcal{I}}$ where

$$\tilde{\mathcal{I}}_i = |\mathcal{I}_i|/\sqrt{T_{ii}}, \quad i = 1, 2, ...n. \tag{6}$$

If $\tilde{\mathcal{I}}_i > \tau$, then $i \in MeasOut$ where $MeasOut$ holds the measurement outlier indices, indicating abnormal measurements.

Now we can separate the measurement "noise" from the process "noise". Let us denote $ProcOut = Out \backslash MeasOut = \{i : i \in Out \quad and \quad i \notin MeasOut\}$ as the set difference between Out and $MeasOut$. If $MeasOut$ is not empty, we will recalculate the ΔQ as in the optimization problem (5), except only for $\forall i \in ProcOut$, and update Q to $Q + \Delta Q$. As for the measurements, we "inflate" R in this way: for $\forall i \in MeasOut$, R_{ii} is inflated to $\lambda_i R_{ii}$. As a result, $T = RS^{-1}R$ is also inflated: the ith diagonal element is now $\lambda_i^2 T_{ii}$ and

$$\tilde{\mathcal{I}}_i = |\mathcal{I}_i|/\lambda_i\sqrt{T_{ii}} \leq \tau, \quad i = 1, 2, ...n. \tag{7}$$

It is very easy to compute λ_is by

$$\lambda_i = (|\mathcal{I}_i|/\sqrt{T_{ii}})/\tau, \quad i \in MeasOut. \tag{8}$$

Finally, with the inflated Q and R, we recompute the correction step to obtain a more robust state estimation. The Q and R are also updated and carried on to the next cycle.

Furthermore, Q and R should also be able to deflate if the abnormal process/measurement problems are only temporary and eventually resolved. Our solution is to employ an exponential decay process to enable automatic deflation of the parameters during each cycle. The decay speed can be customized by users, according to their expectation and the specific circumstances.

In this context, we are interested in power system state estimation in particular. The invention of Phasor Measurement Units (PMUs) has significantly improved its accuracy and frequency: Precisely synchronized from the common global positioning system (GPS) radio clock, PMUs are able to provide high-speed (30Hz or higher) voltage and current phasor measurements [18]. Here we assume that a PMU installed at a specific bus can measure the bus voltage phasor, as well as the current phasors along all the lines incident to the bus. A more detailed system modeling can be found in our previous research [15]. According to [4], the rectangular coordinate formulation will be preferable, where the real and imaginary parts of bus voltages are considered state variables.

Due to the cyber-physical nature of PMUs, the measurements are exposed to all kinds of device failures and data attacks. Taking into account that the system is also subjected to various disturbances, our proposed method will provide valuable insights on such systems. Sometimes with limited budget, we may have to place PMUs selectively in scattered locations to cover the entire system [15]–[17], resulting in a low data redundancy level; however, it can be increased if we treat the state predictions in our dynamic estimation method as another input source.

IV. Case Studies

The proposed approach, AKF with InNoVa, has been simulated on different multi-machine systems under various conditions. For a more realistic and comprehensive study, we will verify the feasibility of our method on the 16-generator-68-bus New England Test System and New York Power System model (Fig. 1) in this section. Simulations are displayed in steps of 0.01 seconds. At $t = 1.1$ second a three-phase fault at bus 29 occurs, and is then cleared at 0.05 seconds. This event represents an unexpected disturbance (an emergency) causing voltage oscillations.

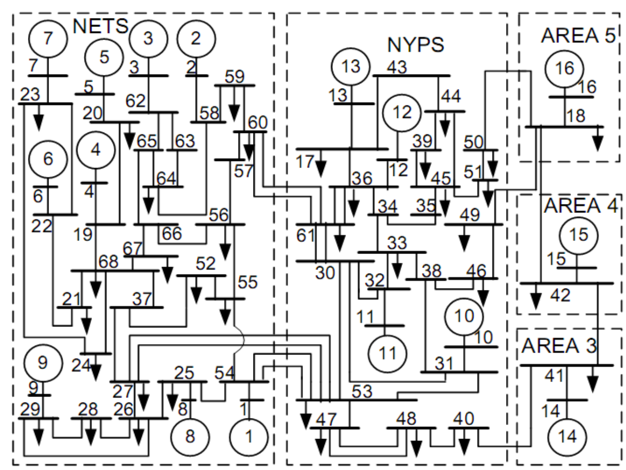

Fig. 1. The 16-generator-68-bus New England Test System and New York Power System model

To make it closer to real-world applications, PMU data are assumed to contain 1% random noise. The latest IEEE Standard for Synchrophasor Measurements for Power Systems, *IEEE C37.118.1-2011* ([18]), has defined an aggregated error quantity called "Total Vector Error" (*TVE*) as the measure of accuracy for PMUs, and established a criterion of 1% for the value of *TVE* during calibration. Normalized and expressed as per unit of the theoretical phasor, *TVE* is calculated in the following way:

$$\text{TVE}_n = \sqrt{\frac{(Re(\hat{x}_n) - Re(x_n))^2 + (Im(\hat{x}_n) - Im(x_n))^2}{(Re(x_n))^2 + (Im(x_n))^2}} \tag{9}$$

where $Re(\hat{x}_n)$ and $Im(\hat{x}_n)$ are the estimated phasor values given by the unit under test at time instant n, $Re(x)_n$ and $Im(x)_n$ are the true phasor values of the input signal at time instant n. Note that as with most measurements, the "true" value can never be precisely known, thus we rely on calibrations to establish the bounds within which the measurement (the vector) has a high probability of occurring.

In the following five subsections, five abnormal cases are simulated to illustrate the robustness of our proposed algorithm. We assume that the users are unaware of any possible contingencies, hence a quasi-static model of a power system has been employed as the process model (Note that the model can be incorrect, *i.e.* sub-optimal, but our algorithm adapts dynamically). For a more informative comparison, in each case we executed three Kalman filters: the traditional Kalman filter (KF), the naive robust KF (RKF) which uses the largest normalized innovation test to identify and exclude bad measurements without adjusting model parameters, and the AKF with InNoVa.

Then the complex voltage Mean Absolute Error and their histogram plots are illustrated for a more comprehensive comparison. Mean Absolute Error, denoted by *MAE*, is one of the most commonly used accuracy evaluation criteria in quantitative methods of forecasting. In this particular application, the states are complex voltages. So at each time step the *MAE* can be written as:

$$MAE = \frac{1}{n}\sum_{i=1}^{n}\|\hat{x}_i - x_i\| = \frac{1}{n}\sum_{i=1}^{n}\|e_i\| \qquad (10)$$

where $\hat{x}_i$ is the estimated state at bus i, x_i is the actual state at bus i, and $e_i = \|\hat{x}_i - x_i\|$ is the absolute error (norm of the complex error). To maintain consistency of our experiments in this subsection, the data range of each histogram is set to be $[0, 0.12]$, and the bin (*i.e.* interval) size is set to be 0.001.

A. Case 1: low process noise and measurement noise setting

We first initialize each state variable in *per unit* with a small process noise variance ($2.5e^{-5}$), meaning that the user assumes the system is very stable. Then we initialize the PMU measurement noise to 1%, but the actual PMU measurements contain 2% noise. In this and the following simulations, we always use a red dash-dash line for the traditional KF, a green dash-dot line for the naive RKF, and a blue dot-dot line for the AKF with InNoVa—our proposed approach.

The accuracy evaluation of all filters can be found in Fig. 2. The small process noise Q setting has dominated the behavior of the KF and the naive RKF, causing them to be overconfident with the quasi-static model, where the system state has slow and small oscillations. Yet by adjusting process noise Q and measurement noise R, the AKF with InNoVa closely tracks the true state. We have observed that bus 29 voltages have the largest process variance (0.1180) which is no surprise because the fault is simulated at bus 29. The average measurement noise has grown to 1.4% in 5 seconds.

B. Case 2: low process noise setting, a PMU under attack

While process noise is initialized same as above, measurement noise is set properly this time, i.e. PMU measurements are simulated to contain 1% noise. We introduce a new problem: a PMU at bus 22 is under attack. An additional random variable error, with normal distribution $\mathcal{N}(1, 0.1^2)$, was injected to the voltage measurements provided by this PMU.

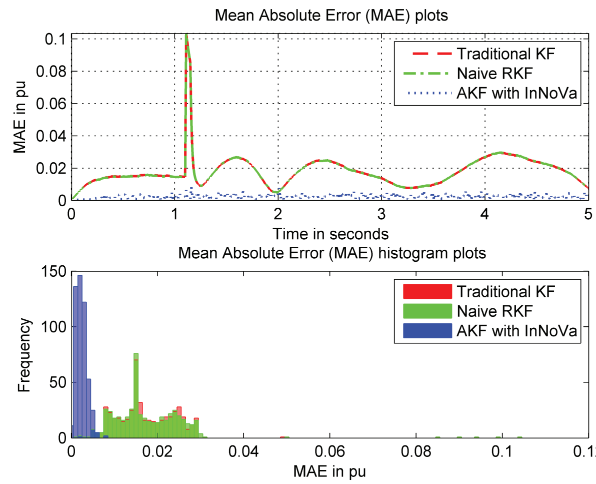

Fig. 2. Voltage estimation accuracy comparison of three filters in case 1

Fig. 3 shows that the naive RKF can identify and drop the bad measurements. However, it is still affected by the under-valued process noise Q. The AKF with InNoVa not only adjusted process noise parameters (bus 29 has process variance grown to 0.1165, which remains the largest), but also identified the bad measurement. Although our algorithm does not exclude the measurement—we seek to avoid unobservable conditions by maintaining sufficient redundancy—the noise of this bad data injected measurement becomes 41.21%, which is significantly larger than the others (1%). Hence this measurement is not weighted as heavily, and has a negligible impact on the estimated states. Fig. 3 also confirms that while the naive RKF does improve, the AKF with InNoVa stays closer to the truth.

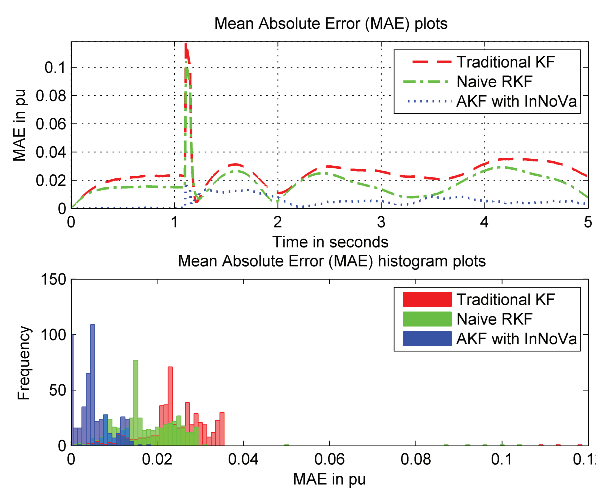

Fig. 3. Voltage estimation accuracy comparison of three filters in case 2

C. Case 3: fair noise setting, a PMU under attack

This time the PMU at bus 22 has the same bad data injection as described in case 2, but the initial process noise variance of each state variable is 0.05, which is more appropriate for our testing system.

In five seconds, the AKF with InNoVa has its process noise variance at bus 29 grown to 0.1166; it also identifies the bad measurements by inflating the corresponding measurement noise level to 41.17%. Compared to others at 1%, it is quite obvious that we are having a bad measurement at bus 22. Fig. 4 tells us that with this fair noise setting, while the AKF with InNoVa still performs the best, the naive RKF is able to correctly detect, identify and eliminate the bad data most of the time. Thus in this case naive RKF can behave quite close to the AKF with InNoVa in this case, while traditional KF produces a shifted tracking result because of the bad PMU data.

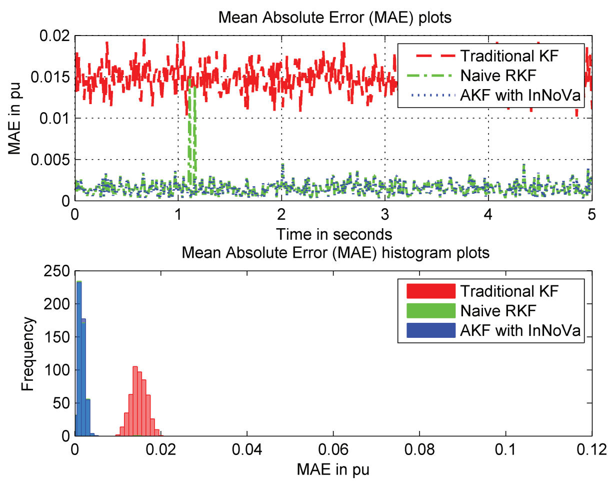

Fig. 4. Voltage estimation accuracy comparison of three filters in case 3

D. Case 4: high process noise setting, a PMU under attack

Now the initial process noise variances are each assigned a much larger value: 1, meaning that the user is not confident with the process model. This time the attacker of the PMU at bus 22 increases the injected error: a random variable with distribution $\mathcal{N}(2.8, 0.1^2)$ is added to the PMU voltage measurements.

Interestingly, MAE plots in Fig. 5 shows that the naive RKF rarely detects the bad data. The reason is that process noise Q is set so large (hence S is so large), the *normalized innovations* almost always fall under the threshold. However the error is always detected by the *normalized residual test* in the AKF with InNoVa. As a matter of fact, in comparison with other measurement noise at 1%, the noise of this bad measurement has been inflated to 100.95%. Thus, as shown in Fig. 5, the AKF with InNoVa always performs the best, while the naive RKF performs as bad as traditional KF almost all the time.

E. Case 5: high process noise setting, multiple locations under attack

Although process noise covariance Q is initialized with the same large values as in case 4, case 5 has more complicated attacks (note that although errors are treated as Gaussian

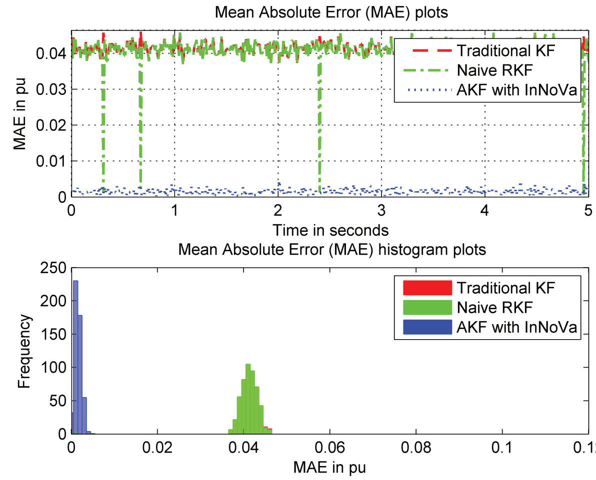

Fig. 5. Voltage estimation accuracy comparison of three filters in case 4

white noise, they do NOT have to be Gaussian and white): at $t = 1$, the attacker increases bus 22 voltage measurements by a constant value of 4 for 0.1 seconds; at $t = 2$, a normally distributed random variable $\mathcal{N}(0, 5^2)$ is added to bus 23 voltage measurements for 0.2 seconds; at $t = 3$, a uniformly distributed random variable $\mathcal{U}(0, 10)$ is added to line $22 - 21$ current measurements for 0.3 seconds; at $t = 4$, the attacker decreases line $23 - 22$ current measurements imaginary-part by a constant value of 15 for 0.4 seconds.

It is clear that traditional KF is the most vulnerable to these disturbances. When the disturbance is large enough (e.g., $t = 1$), the naive RKF is capable of detecting the bad measurements and calibrating the estimates, but otherwise it underperforms. Fortunately, the AKF with InNoVa remains robust throughout. Moreover, as the corresponding measurement noise variances suddenly become abnormally large, the inflated measurement noise R can be used to signal humans of a need to inspect and repair devices or defend against attacks. Fig. 6 shows that various noise disturbances have neglectable effect on the AKF with InNoVa. Therefore, the AKF with InNoVa still outputs the most impressive state estimating results with the highest accuracy level.

V. Conclusions and Future Work

This paper presents a novel Kalman filtering technique: adaptive Kalman filter with inflatable noise variances (AKF with InNoVa). With real-time phasor measurements provided by PMUs, this algorithm enables secure and robust power system state estimation under various adverse conditions, given that there is sufficient redundancy among measurements. It is capable of doing so because in this approach, besides the regular *normalized innovation test*, we also employ a *normalized residual test* to help separate the process and measurement factors.

Designed to deal with incorrect system modeling as well as bad measurements, this algorithm is able to adjust the noise modeling parameters on-the-fly. More specifically, the inflation of process noise covariance Q indicates fast changing state

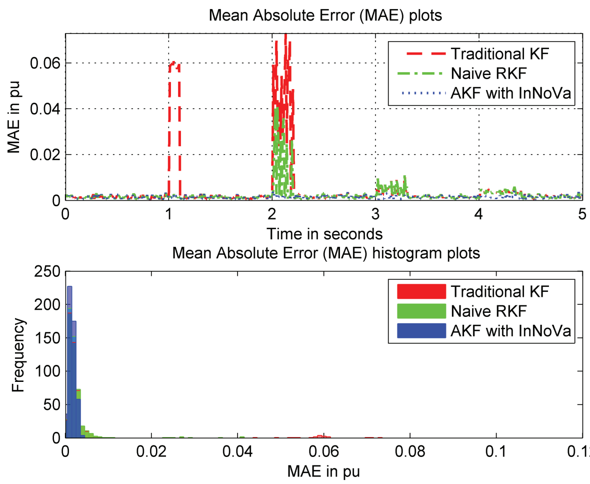

Fig. 6. Voltage estimation accuracy comparison of three filters in case 5

or even wrong model, while the inflation of measurement noise covariance R implies potentially bad measurements. Furthermore, an exponential decay process is employed to enable automatic deflation of the parameters if the problems are resolved.

With the advance of *big data* technologies, "patterns" and "signatures" of bad data can be analyzed and extracted from the massive historical data. A future research direction could be the development of more sophisticated identification methods to categorize bad data into finer classes. Also, the application of our algorithm in strategically defending against the cyber and physical attacks in smart grid will be further explored.

Appendix A
Proof of Lemma 1

Proof: According to the *a posteriori* state estimate in (1) by incorporating the measurement, we have

$$\begin{aligned}\hat{x}_k &= \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)\\ H\hat{x}_k &= H\hat{x}_k^- + HK_k(z_k - H\hat{x}_k^-)\\ z_k - H\hat{x}_k &= (z_k - H\hat{x}_k^-) - HK_k(z_k - H\hat{x}_k^-)\\ z_k - H\hat{x}_k &= (I - HK_k)(z_k - H\hat{x}_k^-),\end{aligned} \tag{11}$$

where I is the identity matrix. Then the mean of the *residual* is

$$E(z_k - H\hat{x}_k) = (I - HK_k)E(z_k - H\hat{x}_k^-) = 0, \tag{12}$$

and the covariance of the *residual* is

$$\begin{aligned}cov(z_k - H\hat{x}_k) &= (I - HK_k)cov(z_k - H\hat{x}_k^-)(I - HK_k)^T\\ &= (I - HK_k)S_k(I - HK_k)^T.\end{aligned} \tag{13}$$

Next we will show that $I - HK_k = RS^{-1}$:

$$\begin{aligned}I - HK_k &= I - HP_k^- H^T(HP_k^- H^T + R)^{-1}\\ &= R(HP_k^- H^T + R)^{-1}\\ &= RS_k^{-1}.\end{aligned} \tag{14}$$

Because the *innovation covariance* S_k and measurement noise covariance R are both symmetric matrices, by combining (13) and (14) we can now write

$$\begin{aligned}cov(z_k - H\hat{x}_k) &= RS_k^{-1}S_k(RS_k^{-1})^T\\ &= R(S_k^{-1})^T R^T\\ &= RS_k^{-1}R.\end{aligned} \tag{15}$$

Thus $T_k = RS_k^{-1}R$ is the *residual covariance*.

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